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Studying Three Types of Matrix Fractional Integrals

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional integral and a new multiplication of fractional analytic functions, we obtain three types of matrix fractional integrals. The matrix fractional exponential function, matrix fractional cosine function, and matrix fractional sine function play important roles in this article. In fact, our results are generalizations of the results in classical calculus.**

Keywords: **Jumarie type of R-L fractional integral, new multiplication, fractional analytic functions, matrix fractional integrals, matrix fractional functions.**

I. INTRODUCTION

Fractional calculus is the theory of derivative and integral of non-integer order, which can be traced back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. Fractional calculus has been attracting the attention of scientists and engineers from long time ago, and has been widely used in physics, mechanics, control theory, viscoelasticity, electrical engineering, biology, economics and other fields [1-13].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [14-17]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we obtain the following three types of matrix fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[E_\alpha(\lambda A x^\alpha)\otimes_\alpha E_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right],$$

$$\left({}_0I_x^{\alpha}\right)\left[E_\alpha(\lambda A x^\alpha)\otimes_\alpha cos_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right],$$

$$\left({}_0I_x^{\alpha}\right)\left[E_\alpha(\lambda A x^\alpha)\otimes_\alpha sin_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right],$$

where $0<\alpha\leq 1$, λ, r, s are real numbers, and A is an invertible matrix. The matrix fractional exponential function, matrix fractional cosine function, and matrix fractional sine function play important roles in this article. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ . \quad (1)$$

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And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([19])**:** *If* α,β,x_0,C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

In the following, the definition of fractional analytic function is introduced.

Definition 2.3 ([20])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha k}. \tag{12}$$

Definition 2.6 ([23])**:** If $0 < \alpha \le 1$, x is a real variable and A is a matrix. The matrix α-fractional exponential function, matrix α-fractional cosine function, and matrix α-fractional sine function are defined as follows:

$$E_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^k \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}, \tag{13}$$

$$cos_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^{2k} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{14}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{k=0}^{\infty} A^{2k+1} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{15}$$

III. MAIN RESULTS

In this section, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we obtain three types of matrix fractional integrals.

Theorem 3.1: *If* $0 < \alpha \le 1$, λ, r, s *are real numbers,* $ks + \lambda \ne 0$ *for all nonnegative integer* k, *and* A *is an invertible matrix, then*

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha)\otimes_\alpha E_\alpha\big(rE_\alpha(sAx^\alpha)\big)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} r^k \frac{1}{ks+\lambda} A^{-1} E_\alpha((ks+\lambda)Ax^\alpha). \tag{16}$$

Proof Since $E_\alpha(\lambda Ax^\alpha)\otimes_\alpha E_\alpha\big(rE_\alpha(sAx^\alpha)\big)$

$$= E_\alpha(\lambda Ax^\alpha)\otimes_\alpha \sum_{k=0}^{\infty} \frac{1}{k!}\big(rE_\alpha(sAx^\alpha)\big)^{\otimes_\alpha k}$$

$$= E_\alpha(\lambda Ax^\alpha)\otimes_\alpha \sum_{k=0}^{\infty} \frac{1}{k!} r^k E_\alpha(ksAx^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} r^k E_\alpha((ks+\lambda)Ax^\alpha).$$

It follows that

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha)\otimes_\alpha E_\alpha\big(rE_\alpha(sAx^\alpha)\big)\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{1}{k!} r^k E_\alpha((ks+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} r^k \left({}_0I_x^\alpha\right)\left[E_\alpha((ks+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} r^k \frac{1}{sk+\lambda} A^{-1} E_\alpha((ks+\lambda)Ax^\alpha).$$

q.e.d.

Theorem 3.2: *If* $0 < \alpha \le 1$, λ, r, s *are real numbers,* $2ks + \lambda \ne 0$ *for all nonnegative integer* k, *and* A *is an invertible matrix, then*

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha)\otimes_\alpha cos_\alpha\big(rE_\alpha(sAx^\alpha)\big)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} \frac{1}{2ks+\lambda} A^{-1} E_\alpha((2ks+\lambda)Ax^\alpha). \tag{17}$$

Proof $E_\alpha(\lambda Ax^\alpha)\otimes_\alpha cos_\alpha\big(rE_\alpha(sAx^\alpha)\big)$

$$= E_\alpha(\lambda Ax^\alpha)\otimes_\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\big(rE_\alpha(sAx^\alpha)\big)^{\otimes_\alpha 2k}$$

$$= E_\alpha(\lambda Ax^\alpha)\otimes_\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} E_\alpha(2ksAx^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} E_\alpha((2ks+\lambda)Ax^\alpha) .$$

Therefore,

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha) \otimes_\alpha cos_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} E_\alpha((2ks+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} \left({}_0I_x^\alpha\right)\left[E_\alpha((2ks+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k} \frac{1}{2ks+\lambda} A^{-1} E_\alpha((2ks+\lambda)Ax^\alpha) .$$

q.e.d.

Theorem 3.3: *If* $0 < \alpha \leq 1$, λ, r, s *are real numbers,* $(2k+1)s + \lambda \neq 0$ *for all nonnegative integer* k, *and* A *is an invertible matrix, then*

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha) \otimes_\alpha sin_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} \frac{1}{(2k+1)s+\lambda} A^{-1} E_\alpha(((2k+1)s+\lambda)Ax^\alpha). \quad (18)$$

Proof Since $E_\alpha(\lambda Ax^\alpha) \otimes_\alpha sin_\alpha\left(rE_\alpha(sAx^\alpha)\right)$

$$= E_\alpha(\lambda Ax^\alpha) \otimes_\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(rE_\alpha(sAx^\alpha)\right)^{\otimes_\alpha (2k+1)}$$

$$= E_\alpha(\lambda Ax^\alpha) \otimes_\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} E_\alpha((2k+1)sAx^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} E_\alpha(((2k+1)s+\lambda)Ax^\alpha) .$$

It follows that

$$\left({}_0I_x^\alpha\right)\left[E_\alpha(\lambda Ax^\alpha) \otimes_\alpha sin_\alpha\left(rE_\alpha(sAx^\alpha)\right)\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} E_\alpha(((2k+1)s+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} \left({}_0I_x^\alpha\right)\left[E_\alpha(((2k+1)s+\lambda)Ax^\alpha)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r^{2k+1} \frac{1}{(2k+1)s+\lambda} A^{-1} E_\alpha(((2k+1)s+\lambda)Ax^\alpha) .$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we obtain three types of matrix fractional integrals. The matrix fractional exponential function, matrix fractional cosine function, and matrix fractional sine function play important roles in this article. In fact, our results are generalizations of the results in traditional calculus. In the future, we will continue to study the problems in engineering mathematics and fractional differential equations by using our methods.

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